

When the hotter cools more quickly: Mpemba effect in granular fluids

Antonio Lasanta¹, Francisco Vega Reyes^{1,2}, Antonio Prados³, and Andrés Santos^{1,2}

¹*Departamento de Física, Universidad de Extremadura, 06071 Badajoz, Spain*

²*Instituto de Computación Científica Avanzada (ICCAEx),
Universidad de Extremadura, 06071 Badajoz, Spain*

³*Física Teórica, Universidad de Sevilla, Apartado de Correos 1065, E-41080, Sevilla, Spain*

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Under certain conditions, two samples of fluid at initial different temperatures present a counter-intuitive behavior, known as the Mpemba effect: it is the hotter system that cools sooner. Here, we show that the Mpemba effect is present in granular fluids, both in the uniformly heated and in the freely cooling systems. In both cases, the system remains homogeneous and no phase transition is present. Analytical quantitative predictions are given for how differently the system must be initially prepared to observe the Mpemba effect, the theoretical predictions being confirmed by computer simulations. Possible implications of our analysis for other systems are also discussed.

Let us consider two beakers of water at two different temperatures T_h and T_c , respectively, with $T_c < T_h$. If put in contact with a thermal reservoir at sub-zero (in the Celsius scale) temperature, which one freezes first? The intuitive answer is that the initially cooler sample freezes first but experience tells us that this is not always the case. Known for a long time, this effect has even been part of the “popular” culture in cold countries. Moreover, already in 350 BC Aristotle stated that [1] “The fact that the water has previously been warmed contributes to its freezing quickly; for so it cools sooner.”

Physicists only started to analyze this paradoxical behavior in the second part of the past century. In fact, it is named after Erasto B. Mpemba who, together with O. Osborne, conducted its first systematic experimental investigation [2]. Afterwards, as far as we know, the Mpemba effect (ME) has been scarcely studied, considering it mainly as a physical curiosity in non-specialized journals [3–13], with no consensus for a general explanation. See [14, 15] for reviews.

The “classical” Mpemba setup involves the freezing of a liquid in contact with a thermal reservoir [2]. Many aspects of the experiment, such as evaporation [3, 6, 16, 17], differences in the gas composition of water [11], natural convection [18], or the influence of supercooling (alone [8, 19] or combined with other causes [20, 21]), seem to have an impact on the ME. Very recently, the underlying mechanisms have also started to be analyzed by means of molecular dynamics simulations [22]. Moreover, Mpemba-like effects have been observed in carbon nanotube resonators [23] and clathrate hydrates [24].

The ME shows that the evolution equation for the temperature involves other variables, which may facilitate or hinder the temperature relaxation rate. The initial values of those additional variables depend on the way the system has been prepared (i.e., “aged”), before starting the relaxation process. Typically, aging and memory effects are associated with slowly evolving systems with a complex energy landscape, such as glassy [25–32] or dense granular systems [33–35]. Notwithstanding, these effects

have also been observed in systems with a very simple energy landscape, like granular gases [36–39].

Considering the ME in a general physical system, some relevant questions arise: What are those additional variables that control the temperature relaxation? How different do they have to be initially in order to facilitate the emergence of the ME? In order to analyze the ME with the tools of nonequilibrium statistical mechanics, a more precise definition thereof, which makes it possible to investigate it quantitatively, is mandatory. An option is to consider the relaxation time to the final temperature as a function of the initial temperature [2, 4, 6, 7, 14, 17, 24]. Alternatively, one may look at the time-dependent relaxation curves of the temperature: if the curve for the initially hotter system crosses that of the initially cooler one and remains below it for longer times, the ME is present [6, 12, 15, 17, 19–21, 23].

In this Letter, we investigate the ME in a prototypical case of intrinsically out-of-equilibrium system: a granular fluid [40–42], i.e., a (dilute or moderately dense) set of mesoscopic particles that do not preserve energy upon collision. As a consequence, the mean kinetic energy, or granular temperature $T(t)$, decays monotonically in time unless an external energy input is applied. The simplicity of the granular fluid makes it an ideal benchmark for other, more complex, nonequilibrium systems. Here, the homogeneous heated and freely cooling cases are analyzed.

We show that the ME naturally appears as a consequence of the relevance of non-Gaussianities in the time evolution of T . Specifically, this allows us to (i) show that the ME is to be expected on quite a general basis and for a wide range of systems and (ii) quantitatively predict the region of parameters for which the effect is present. Moreover, we observe an *inverse* ME when the system is heated instead of cooled [43]: the initially cooler sample heats sooner. Thus, we provide a general theoretical framework for the understanding of the ME. What is more, our analytical theoretical predictions are checked against numerical simulations of the

inelastic Enskog equation, showing an excellent agreement.

Let us consider for our granular fluid the smooth hard-sphere collisional model. Therein, the component of the relative velocity along the line joining the centers of the two colliding particles is reversed and (due to inelasticity) shrunk by a factor α , the so-called coefficient of normal restitution. In addition, the particles are subject to random forces in the form of a white-noise thermostat with variance $m^2\xi^2$, where m is the mass of a particle. Therefore, the velocity distribution function (VDF) $f(\mathbf{v}, t)$ obeys an Enskog-Fokker-Planck kinetic equation [44, 45].

The granular temperature is defined as $T(t) = \frac{m}{3}\langle v^2(t) \rangle \equiv \frac{m}{3n} \int d\mathbf{v} v^2 f(\mathbf{v}, t)$, where $n = \int d\mathbf{v} f(\mathbf{v}, t)$ is the number density. Its time evolution is coupled to that of the excess kurtosis (or second Sonine coefficient) $a_2 = \frac{3}{5}\langle v^4 \rangle / \langle v^2 \rangle^2 - 1$. From the kinetic equation for the VDF one readily finds

$$\frac{dT}{dt} = -\frac{2K}{3} (\mu_2 T^{3/2} - \chi), \quad (1a)$$

$$\frac{d \ln(1 + a_2)}{dt} = \frac{4K}{3T} \left(\mu_2 T^{3/2} - \chi - \frac{\frac{1}{5}\mu_4 T^{3/2} - \chi}{1 + a_2} \right), \quad (1b)$$

where $K = 2ng(n)\sigma^2\sqrt{\pi/m}$, σ and $g(n)$ being the sphere diameter and the pair correlation function at contact [46], respectively, $\chi = \frac{3}{2}m\xi^2/K$, and μ_2 and μ_4 are dimensionless collisional rates [44]. In order to close the system of exact equations (1), we need to express the collisional rates as functions of a_2 . Thus, we consider the first Sonine approximation, which yields [44] $\mu_n \simeq \mu_n^{(0)} + \mu_n^{(1)} a_2$, with $\mu_2^{(0)} = 1 - \alpha^2$, $\mu_2^{(1)} = \frac{3}{16}\mu_2^{(0)}$, $\mu_4^{(0)} = (\frac{9}{2} + \alpha^2)\mu_2^{(0)}$, $\mu_4^{(1)} = (1 + \alpha) [2 + \frac{3}{32}(69 + 10\alpha^2)(1 - \alpha)]$.

In the long time limit, the uniformly heated granular fluid reaches a steady state where the rates of energy loss (in collisions) and input (by the white-noise force) are balanced. The stationary values of the temperature and the excess kurtosis are obtained by seeking the time-independent solution of Eqs. (1), with the result, in the first Sonine approximation, $T_s = (\chi/\mu_2^s)^{\frac{2}{3}}$ and $a_2^s = 16(1 - \alpha)(1 - 2\alpha^2)/[241 - 177\alpha + 30\alpha^2(1 - \alpha)]$ [44, 45]. The steady collisional moments are simply given by $\mu_n^s = \mu_n^{(0)} + \mu_n^{(1)} a_2^s$.

The evolution equation (1a) for the temperature is not closed because the dimensionless cooling rate μ_2 depends on (in fact, increases with) the excess kurtosis a_2 . Therefore, the relaxation of the granular temperature T to its stationary value from an initially “cooler” (smaller T) sample can be overtaken by that of an initially “hotter” one, if the latter has also a larger enough value of the excess kurtosis initially. We build on and quantify the implications of this physical idea in the following.

First, we prepare the granular fluid in an initial state

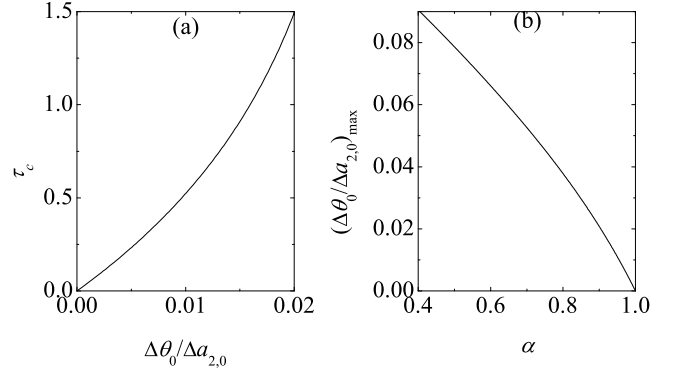


Figure 1. (a) Crossing time τ_c as a function of the ratio $\Delta\theta_0/\Delta a_{2,0}$ for $\alpha = 0.9$. (b) $(\Delta\theta_0/\Delta a_{2,0})_{\max}$ as a function of the coefficient of restitution α .

that is close to the steady one, in the sense that Eqs. (1) can be linearized around (T_s, a_2^s) . Let us use a dimensionless temperature $\theta = T/T_s$ and define $\delta\theta = \theta - 1$, $\delta a_2 = a_2 - a_2^s$, and $\tau = K T_s^{1/2} t$. A straightforward calculation gives

$$\frac{d}{d\tau} \begin{pmatrix} \delta\theta \\ \delta a_2 \end{pmatrix} = -\Lambda \cdot \begin{pmatrix} \delta\theta \\ \delta a_2 \end{pmatrix}, \quad (2)$$

where the matrix Λ has elements $\Lambda_{11} = \mu_2^s$, $\Lambda_{12} = \frac{2}{3}\mu_2^{(1)}$, $\Lambda_{21} = -2\mu_2^s a_2^s$, and $\Lambda_{22} = \frac{4}{15}[\mu_4^{(1)} - 5\mu_2^{(1)}(1 + a_2^s)]$. Thus, the relaxation of the temperature reads

$$\delta\theta = \frac{1}{\gamma} \left[(\lambda_+ - \mu_2^s) \delta\theta_0 - \frac{2}{3} \mu_2^{(1)} \delta a_{2,0} \right] e^{-\lambda_- \tau} - \frac{1}{\gamma} \left[(\lambda_- - \mu_2^s) \delta\theta_0 - \frac{2}{3} \mu_2^{(1)} \delta a_{2,0} \right] e^{-\lambda_+ \tau}, \quad (3)$$

where $\lambda_{\pm} = \frac{1}{2} [\Lambda_{11} + \Lambda_{22} \pm \sqrt{(\Lambda_{11} - \Lambda_{22})^2 + 4\Lambda_{12}\Lambda_{21}}]$ are the eigenvalues of the matrix Λ and $\gamma \equiv \lambda_+ - \lambda_- > 0$.

Let us consider two initial states $(\theta_0, a_{2,0}) = (\theta_A, a_{2A})$ and (θ_B, a_{2B}) , with $\theta_A > \theta_B$, $a_{2A} > a_{2B}$. Both cooling ($\theta_A > \theta_B > 1$) and heating ($\theta_B < \theta_A < 1$) processes may be considered. From the analytical solution (3), the time τ_c for the possible crossing of the two relaxation curves satisfies

$$\tau_c = \frac{1}{\gamma} \ln \frac{2\mu_2^{(1)} - 3(\lambda_- - \mu_2^s)\Delta\theta_0/\Delta a_{2,0}}{2\mu_2^{(1)} - 3(\lambda_+ - \mu_2^s)\Delta\theta_0/\Delta a_{2,0}}, \quad (4)$$

in which $\Delta\theta_0 = \theta_A - \theta_B$ and $\Delta a_{2,0} = a_{2A} - a_{2B}$. Figure 1(a) displays τ_c as a function of the ratio $\Delta\theta_0/\Delta a_{2,0}$ for $\alpha = 0.9$. Equation (4) implies that there is a maximum of the initial increments ratio, $\Delta\theta_0/\Delta a_{2,0}$, for which the ME can be observed, namely

$$\left(\frac{\Delta\theta_0}{\Delta a_{2,0}} \right)_{\max} = \frac{2}{3} \frac{\mu_2^{(1)}}{\lambda_+ - \mu_2^s}. \quad (5)$$

Figure 1(b) clearly shows that $(\Delta\theta_0/\Delta a_{2,0})_{\max}$ increases with the inelasticity $1 - \alpha$.

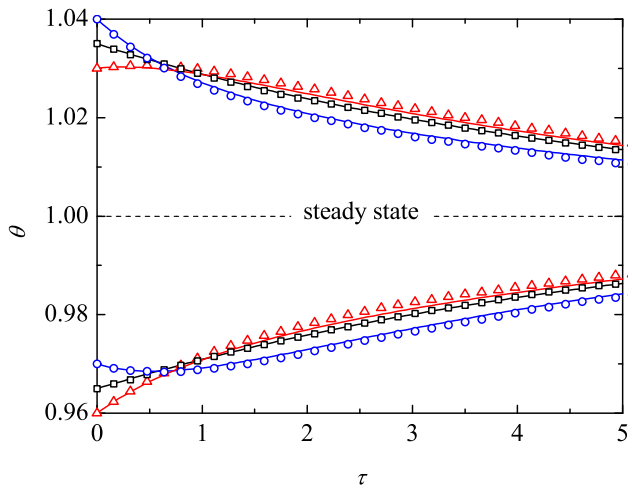


Figure 2. Relaxation of the scaled temperature to the steady state for $\alpha = 0.9$. The upper and the lower curves correspond to the ME for the cooling and heating processes, respectively (see text). The DSMC data (symbols) show an excellent agreement with the theoretical prediction (lines), especially in the early stage.

Equation (5) can be read in two alternative ways. First, it means that, for a given difference $\Delta a_{2,0}$ of the initial kurtosis, the ME appears when the difference $\Delta\theta_0$ of the scaled initial temperatures is below a maximum value $(\Delta\theta_0)_{\max}$, proportional to $\Delta a_{2,0}$. Second, and quite interestingly, for a given value of $\Delta\theta_0$, the ME is observed only for a large enough difference of the initial kurtosis, that is, $\Delta a_{2,0} > (\Delta a_{2,0})_{\min}$, with $(\Delta a_{2,0})_{\min}$ proportional to $\Delta\theta_0$. This quantitatively measures how different the initial preparations of the system must be in order to have the ME.

In order to check the accuracy of our theoretical results, we compare them in Fig. 2 with the numerical integration of the Enskog–Fokker–Planck equation for the granular fluid [45]. We used for this the direct simulation Monte Carlo (DSMC) method [47]. In all our numerical simulations, the initial VDF is assumed to have a gamma-distribution form [48] in the variable v^2 with parameters adjusted to the chosen values of θ and $a_{2,0}$. Specifically, we consider the relaxation curves of the granular temperature to its stationary value for $\alpha = 0.9$. First, three different initial conditions, A, B, and C, with temperatures above the stationary, $\theta_A = 1.04$, $\theta_B = 1.035$, and $\theta_C = 1.03$, and excess kurtosis $a_{2A} = 0.5$, $a_{2B} = 0$, and $a_{2C} = -0.35$, are considered. The ME is clearly observed as a crossing of the relaxation curves of the temperature [see also Fig. 1(a)]. Second, we analyze a “heating” protocol, by considering initial temperatures below the steady value, namely $\theta'_A = 0.97$, $\theta'_B = 0.965$, and $\theta'_C = 0.96$, with the same values of the excess kurtosis as in the “cooling” case. Again, a crossing in the temperature relaxation curves appears, signaling the granular analog of the *inverse* ME proposed in a recent work [43]. It is in-

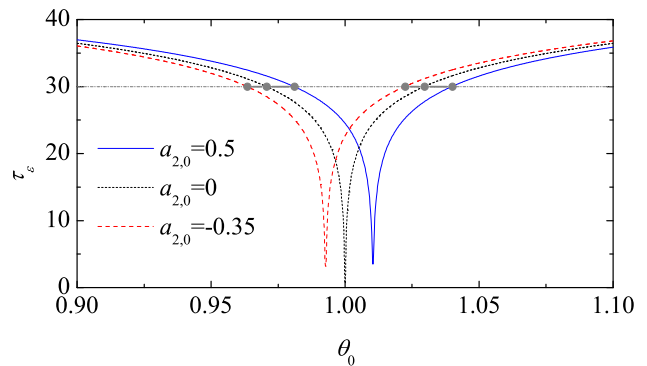


Figure 3. Relaxation time τ_ϵ (with $\epsilon = 10^{-4}$) as a function of the initial scaled temperature θ_0 for $\alpha = 0.9$. Three values of the initial excess kurtosis are considered, $a_{2,0} = 0.5$ (solid line), $a_{2,0} = 0$ (dotted line), and $a_{2,0} = -0.35$ (dashed line). The horizontal (grey) segments join values of initial temperatures that share the same value of the relaxation time and thus mark the onset of either the ME ($\theta_0 > 1$) or the inverse ME ($\theta_0 < 1$).

teresting to note that the evolution curves corresponding to $\theta_C = 1.03$ and $\theta'_A = 0.97$ are non-monotonic. This peculiar behavior is predicted by Eq. (3) to take place if $-\frac{2}{3}\mu_2^{(1)}/\mu_2^s < \delta\theta_0/\delta a_{2,0} < 0$.

Alternatively, we can introduce a relaxation time characterizing the system celerity for cooling (or heating). Given a certain small distance $\epsilon \ll 1$, the relaxation time τ_ϵ is defined as the time beyond which $|\theta(\tau_\epsilon) - 1| < \epsilon$. From Eq. (3), our analytical prediction for τ_ϵ is

$$\tau_\epsilon = \frac{1}{\lambda_-} \ln \left| \frac{3(\lambda_+ - \mu_2^s)\delta\theta_0 - 2\mu_2^{(1)}\delta a_{2,0}}{3\epsilon\gamma} \right|. \quad (6)$$

Figure 3 shows τ_ϵ (with $\epsilon = 10^{-4}$) as a function of the initial temperature θ_0 for the same values of the initial excess kurtosis considered in Fig. 2. In this diagram, for a given pair of $a_{2,0}$, the range of initial temperatures for which the ME emerges is clearly visualized. Note that this range does not change if the value of the bound ϵ is changed to ϵ' , since the diagram is only shifted vertically by an amount $\frac{1}{\lambda_-} \ln(\epsilon/\epsilon')$.

Does the ME survive in the limit of zero driving? In fact, this is a really relevant question in the realm of granular systems, since the undriven granular fluid relaxes to the so-called homogeneous cooling state (HCS), which is the reference state for deriving the granular hydrodynamics [49]. If the linear relaxation picture developed above remained valid in the nonlinear relaxation regime, at least qualitatively, the answer would be negative: if $\chi \rightarrow 0$ ($T_s \propto \chi^{2/3} \rightarrow 0$), the maximum temperature difference $(\Delta T_0)_{\max}$ would also vanish [note that $(\Delta\theta_0)_{\max}$ does not depend on χ]. Interestingly, we show below that this simple scenario does not hold and the ME is also observed for very small driving: in fact, $(\Delta T_0)_{\max}$ remains finite in this limit.

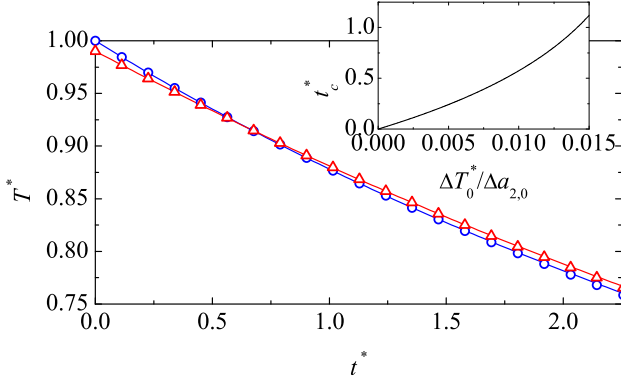


Figure 4. Evolution of the temperature in the free cooling case. Again, the agreement between the DSMC data (symbols) and the theory (lines) is excellent. The inset shows t_c^* as a function of $\Delta T_0^*/\Delta a_{2,0}$.

For very small driving, there is a wide time region inside which the system evolves as if it were cooling freely. Therefore, for the sake of simplicity, we take $\xi = 0$ and thus the system freely cools for all times, and $\lim_{t \rightarrow \infty} T = 0$. In that case, the evolution equations for the temperature and the excess kurtosis are the particularization of Eqs. (1) for $\chi = 0$. Since there is no natural temperature scale in the free cooling case, we can go to dimensionless variables by scaling temperature and time with an arbitrary reference value T_{ref} , i.e., $T^* = T/T_{\text{ref}}$ and $t^* = KT_{\text{ref}}^{1/2}t$. As time increases, the excess kurtosis tends to a constant value $a_2^{\text{HCS}} = 16(1-\alpha)(1-2\alpha^2)/[97-33\alpha-2\alpha^2(1-\alpha)]$ [44, 45].

If at all present, we expect the ME to occur for relatively short times; more specifically, before a_2 has relaxed to its stationary value a_2^{HCS} . So as to look for a possible crossing of the cooling curves, we linearize the equations around $T^* = 1$, $\delta T^* = T^* - 1$, by choosing T_{ref} such that the initial temperatures verify $|T_0^* - 1| \ll 1$, and $a_2^{\text{HCS}}, \delta a_2 = a_2 - a_2^{\text{HCS}}$. Therefrom, the evolution of T^* is obtained as

$$\delta T^* = \left(\delta T_0^* + \frac{2}{3} - \frac{2}{3} \frac{\mu_2^{(1)} \delta a_{2,0}}{\lambda_a - \mu_2^{\text{HCS}}} \right) e^{-\mu_2^{\text{HCS}} t^*} + \frac{2}{3} \frac{\mu_2^{(1)} \delta a_{2,0}}{\lambda_a - \mu_2^{\text{HCS}}} e^{-\lambda_a t^*} - \frac{2}{3}, \quad (7)$$

in which $\lambda_a = (1+\alpha)[113-49\alpha-34\alpha^2(1-\alpha)]/120$. The excess kurtosis decays exponentially to a_2^{HCS} with a characteristic time λ_a^{-1} .

Similarly to the thermostated case, we consider two initial states $(T_0^*, a_{2,0}) = (T_A^*, a_{2A})$ and (T_B^*, a_{2B}) , with $\Delta T^* = T_A^* - T_B^* > 0$, $\Delta a_2 = a_{2A} - a_{2B} > 0$. Logically, only the cooling case makes sense. In Fig. 4, we plot two relaxation curves of the temperature for $\alpha = 0.9$, with $T_A^* = 1$, $T_B^* = 0.99$, $a_{2A} = 0.5$, $a_{2B} = -0.35$, with the choice $T_{\text{ref}} = T_A$. The ME is clearly observed, and the

crossing time t_c^* (see inset in Fig. 4) is

$$t_c^* = \frac{1}{\lambda_a - \mu_2^{\text{HCS}}} \ln \left(1 - \frac{3}{2} \frac{\lambda_a - \mu_2^{\text{HCS}}}{\mu_2^{(1)}} \frac{\Delta T^*}{\Delta a_2} \right)^{-1}. \quad (8)$$

Therefore, there is a maximum value of the ratio $\Delta T^*/\Delta a_2$ for which the ME appears,

$$\left(\frac{\Delta T^*}{\Delta a_2} \right)_{\text{max}} = \frac{2}{3} \frac{\mu_2^{(1)}}{\lambda_a - \mu_2^{\text{HCS}}}. \quad (9)$$

Thus, the ME actually survives in the zero driving limit. Had we considered a small value of the driving χ instead of $\chi = 0$, Eqs. (7)–(9) would characterize the strongly nonlinear regime, in which the initial scaled temperature $\theta_0 = T_0/T_s \gg 1$. In a first stage of the relaxation, as long as the granular temperature $T \gg T_s$, the driving can be neglected, the system freely cools, and the ME is observed provided that the condition (9) is fulfilled. Afterwards, the initially hotter system remains below the initially cooler one forever. When approaching the steady state, both the temperature and the excess kurtosis start to evolve towards their stationary values T_s and a_2^s , but in both curves one has $a_{2,0} = a_2^{\text{HCS}}$ and Eq. (5) tells us that no further crossing of the curves takes place ($\Delta a_{2,0} = 0$).

It must be stressed that the granular fluid remains always in a homogeneous and stable state, neither gradients nor a phase transition are present. In addition, the theory presented here applies to finite density and is based on the first Sonine approximation, in which the non-Gaussianity of the VDF is encoded in only one variable, the excess kurtosis a_2 . However, our quantitative predictions are accurate for the not-so-small values of a_2 considered throughout the paper. Despite these drastic simplifications, the ME (and the inverse ME when heating) is still there.

Our approach is quite general and thus it is not limited to the field of granular fluids. Can a similar approach explain, at least partially, the existence of the ME in a molecular system with elastic collisions? One may argue that the ME found here disappears in the elastic limit $\alpha \rightarrow 1$, since $(\Delta T_0/\Delta a_{20})_{\text{max}} \rightarrow 0$, and, moreover, the *equilibrium* VDF is always a Maxwellian. But, for a system of elastic particles immersed in a fluid producing a *nonlinear* drag [50–52], for instance, its nonequilibrium VDF would also be non-Gaussian and those non-Gaussianities are expected to enter into the evolution equation of the temperature, giving rise to the ME.

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- [1] Aristotle, *The Complete Works of Aristotle: The Revised Oxford Translation*, edited by J. Barnes, Bollingen Series LXXI-2, Vol. 1 (Princeton University Press, Princeton, NJ, 1984) p. 1247.
- [2] E. B. Mpemba and D. G. Osborne, *Phys. Educ.* **4**, 172 (1969).
- [3] G. S. Kell, *Am. J. Phys.* **37**, 564 (1969).
- [4] I. Firth, *Phys. Educ.* **6**, 32 (1971).
- [5] E. Deeson, *Phys. Educ.* **6**, 42 (1971).
- [6] J. Walker, *Sci. Am.* **237**, 246 (1977).
- [7] M. Freeman, *Phys. Educ.* **14**, 417 (1979).
- [8] D. Auerbach, *Am. J. Phys.* **63**, 882 (1995).
- [9] C. Knight, *Am. J. Phys.* **64**, 524 (1996).
- [10] J.-M. Courty and E. Kierlik, *Pour la Science* **342**, 98 (2006).
- [11] J. I. Katz, *Am. J. Phys.* **77**, 27 (2009).
- [12] C. Q. Sun, *Temperature* **2**, 38 (2015).
- [13] A. A. Romanovsky, *Temperature* **2**, 63 (2015).
- [14] M. Jeng, *Am. J. Phys.* **74**, 514 (2006).
- [15] J. D. Brownridge, *Am. J. Phys.* **79**, 78 (2011).
- [16] M. Vynnycky and S. L. Mitchell, *Heat Mass Transf.* **46**, 881 (2010).
- [17] M. Vynnycky and N. Maeno, *Int. J. Heat Mass Transf.* **55**, 7297 (2012).
- [18] P. K. Maciejewski, *J. Heat Transf.* **118**, 65 (1996).
- [19] X. Zhang, Y. Huang, Z. Ma, Y. Zhou, J. Zhou, W. Zheng, Q. Jiang, and C. Q. Sun, *Phys. Chem. Chem. Phys.* **16**, 22995 (2014).
- [20] S. Esposito, R. De Risi, and L. Somma, *Physica A* **387**, 757 (2008).
- [21] M. Vynnycky and S. Kimura, *Int. J. Heat Mass Transf.* **80**, 243 (2015).
- [22] J. Jin and W. A. Goddard, *J. Phys. Chem. C* **119**, 2622 (2015).
- [23] P. A. Greaney, G. Lani, G. Cicero, and J. C. Grossman, *Metall. Mater. Trans. A* **42**, 3907 (2011).
- [24] Y.-H. Ahn, H. Kang, D.-Y. Koh, and H. Lee, *Korean J. Chem. Eng.* **33**, 1903 (2016).
- [25] A. J. Kovacs, J. J. Aklonis, J. M. Hutchinson, and A. R. Ramos, *J. Polym. Sci. Pol. Phys.* **17**, 1097 (1979).
- [26] J.-P. Bouchaud, *J. Phys. I* **2**, 1705 (1992).
- [27] A. Prados, J. J. Brey, and B. Sánchez-Rey, *EPL* **40**, 13 (1997).
- [28] L. L. Bonilla, F. G. Padilla, and F. Ritort, *Physica A* **250**, 315 (1998).
- [29] L. Berthier and J.-P. Bouchaud, *Phys. Rev. B* **66**, 054404 (2002).
- [30] S. Mossa and F. Sciortino, *Phys. Rev. Lett.* **92**, 045504 (2004).
- [31] G. Aquino, A. Allahverdyan, and T. M. Nieuwenhuizen, *Phys. Rev. Lett.* **101**, 015901 (2008).
- [32] A. Prados and J. J. Brey, *J. Stat. Mech.-Theory Exp.* **2010**, P02009 (2010).
- [33] M. Nicodemi and A. Coniglio, *Phys. Rev. Lett.* **82**, 916 (1999).
- [34] C. Josserand, A. V. Tkachenko, D. M. Mueth, and H. M. Jaeger, *Phys. Rev. Lett.* **85**, 3632 (2000).
- [35] J. J. Brey and A. Prados, *Phys. Rev. E* **63**, 061301 (2001).
- [36] S. R. Ahmad and S. Puri, *Phys. Rev. E* **75**, 031302 (2007).
- [37] J. J. Brey, A. Prados, M. G. de Soria, and P. Maynar, *J. Phys. A: Math. Theor.* **40**, 14331 (2007).
- [38] A. Prados and E. Trizac, *Phys. Rev. Lett.* **112** (2014).
- [39] E. Trizac and A. Prados, *Phys. Rev. E* **90** (2014).
- [40] P. K. Haff, *J. Fluid Mech.* **134**, 401 (1983).
- [41] I. Goldhirsch, *Annu. Rev. Fluid Mech.* **35**, 267 (2003).
- [42] T. Pöschel and S. Luding, eds., *Granular Gases*, Lecture Notes in Physics, Vol. 564 (Springer, Berlin, 2001).
- [43] Z. Lu and O. Raz, *arXiv preprint arXiv:1609.05271* (2016).
- [44] T. P. C. van Noije and M. H. Ernst, *Granul. Matter* **1**, 57 (1998).
- [45] J. M. Montanero and A. Santos, *Granul. Matter* **2**, 53 (2000).
- [46] N. F. Carnahan and K. E. Starling, *J. Chem. Phys.* **51**, 635 (1969).
- [47] G. A. Bird, *Molecular Gas Dynamics and the Direct Simulation of Gas Flows* (Clarendon, Oxford, 1994).
- [48] R. V. Hogg and A. T. Craig, *Introduction to Mathematical Statistics*, 4th ed. (Macmillan Publishing, New York, NY, 1978) pp. 103–108.
- [49] J. J. Brey, J. W. Dufty, C. S. Kim, and A. Santos, *Phys. Rev. E* **58**, 4638 (1998).
- [50] J. E. Stout, S. P. Arya, and E. L. Genikhovich, *J. Atmos. Sci.* **52**, 3836 (1995).
- [51] J. Fung, *J. Geophys. Res.* **103**, 27 905 (1998).
- [52] F. Maggi, *J. Geophys. Res. Oceans* **118**, 2118 (2013).